SAT in AI: high performance search methods with applications

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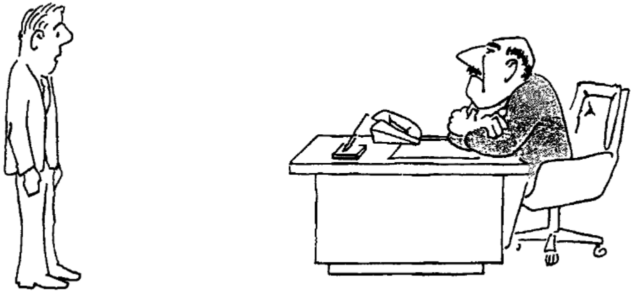
SAT and NP-complete problems in Artificial Intelligence

- Earlier, NP-complete problems were considered practically unsolvable, except in simplest instances.
- Breakthroughs in SAT solving from mid-1990's on.
- Leading to breakthroughs in state space search (with applications in construction of intelligent systems.)
- Starting to have impact in other areas, including probabilistic reasoning and machine learning.

Why you need*ed* to know about NP-hardness

Garey & Johnson, *Computers and Intractability*, 1979



“I can’t find an efficient algorithm, I guess I’m just too dumb.”

Why you need*ed* to know about NP-hardness

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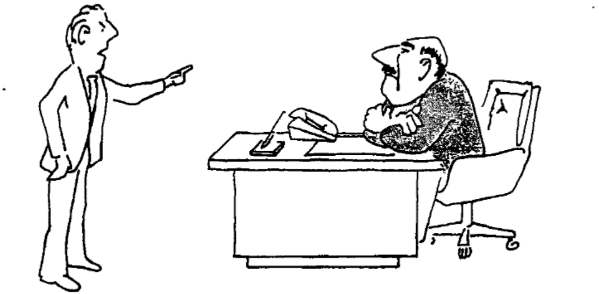
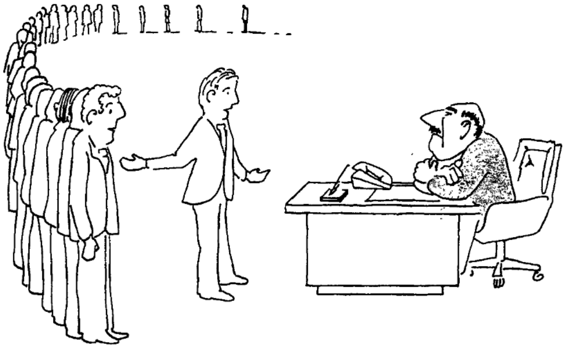


“I can’t find an efficient algorithm, because no such algorithm is possible!”

Why you needed *ed* to know about NP-hardness

Garey & Johnson, *Computers and Intractability*, 1979



"I can't find an efficient algorithm, but neither can all these famous people."

NP-completeness has changed

- Earlier: “It is NP-complete, don’t bother trying to solve it.”
- Now: “It is NP-complete, you might well solve it.”
- SAT now has several industrial applications, and more are emerging.
- Extensions of SAT are a topic of intense research in automated reasoning and AI.
- Many important problems in AI and CS are NP-complete:
  - Combinatorics of the real world (too many options to do things).
  - How to do something optimally?

Applications of SAT in Computer Science

- reachability problems
  - model-checking in Computer Aided Verification [BCCZ99] of sequential circuits and software
  - planning in Artificial Intelligence [KS92, KS96]
  - discrete event systems diagnosis [GARK07]
- integrated circuits
  - automatic test pattern generation (ATPG) [Lar92]
  - equivalence checking [KPKG02, CGL^{+}10, WGMD09]
  - logic synthesis [KKY04]
  - fault diagnosis [SVFAV05]
- biology and language
  - haplotype inference [LMS06]
  - computing evolutionary tree measures [BSJ09]
  - construction of phylogenetic trees [BEE^{+}07]

Classification of Problems by Complexity

problem		class	search space
SAT	find a solution	NP	trees
SMT	find a solution	NP	
MAX-SAT	find best solution	FP^{NP}	
#SAT	how many solutions?	#P, PP	
SSAT	$\exists - \forall - R$ alternation	PSPACE	and-or trees
QBF	$\exists - \forall$ alternation	PSPACE	

Differences in NP-hardness

Most scalable methods are for satisfiable instances of SAT (NP). These can be solved because of good heuristics: solvers are successfully guessing their way through an exponentially large search space.

Currently, the same does not (as often) hold for

- unsatisfiable instances: determining that no solutions exist
- optimization: finding best solutions
- problems involving counting models, e.g. probabilistic questions
- problems involving alternation $\sim$ and-or trees

Progress in these questions is made, but it has been slower.
NP substantially easier than co-NP, #P, FP^{NP}, ...

Propositional logic
Syntax

Let X be a set of atomic propositions.

1. $\bot$ and $\top$ are formulae.
2. x is a formula for all $x \in X$.
3. $\neg\phi$ is a formula if ϕ is.
4. $\phi \vee \phi'$ and $\phi \wedge \phi'$ are formulae if ϕ and ϕ' are.

$\phi \rightarrow \phi'$ is an abbreviation for $\neg\phi \vee \phi'$.
$\phi \leftrightarrow \phi'$ is an abbreviation for $(\phi \rightarrow \phi') \wedge (\phi' \rightarrow \phi)$.

For literals $l \in X \cup \{\neg x | x \in X\}$, complement $\bar{l}$ is defined by $\overline{x} = \neg x$ and $\overline{\neg x} = x$.

A clause is a disjunction of literals $l_1 \vee \cdots \vee l_n$.

Propositional logic
Valuations and truth

Define truth with respect to a valuation $v : X \to \{0, 1\}$:

1. $v \models \top$
2. $v \not\models \bot$
3. $v \models x$ if and only if $v(x) = 1$, for all $x \in X$.
4. $v \models \neg\phi$ if and only if $v \not\models \phi$.
5. $v \models \phi \vee \phi'$ if and only if $v \models \phi$ or $v \models \phi'$.
6. $v \models \phi \wedge \phi'$ if and only if $v \models \phi$ and $v \models \phi'$.

Define for sets C of formulas, $v \models C$ iff $v \models \phi$ for all $\phi \in C$.

The SAT decision problem

SAT

Let X be a set of propositional variables. Let $\mathcal{F}$ be a set of clauses over X.
$\mathcal{F} \in$ SAT iff there is $v : X \to \{0, 1\}$ such that $v \models \mathcal{F}$.

UNSAT

Let X be a set of propositional variables. Let $\mathcal{F}$ be a set of clauses over X.
$\mathcal{F} \in$ UNSAT iff $v \not\models \mathcal{F}$ for all $v : X \to \{0, 1\}$.

Complexity class NP

- NP = decision problems solvable by nondeterministic Turing Machines with a polynomial bound on the number of computation steps.
- This is roughly: search problems with a search tree (OR tree) of polynomial depth.
- SAT is in NP because
  1. a valuation v of X can be guessed in $|X|$ steps, and
  2. testing $v \models \mathcal{F}$ is polynomial time in the size of $\mathcal{F}$.

NP-hardness of SAT
(Cook, *The Complexity of Theorem Proving Procedures*, 1971)

- Cook showed that the halting problem of any nondeterministic Turing machine with a polynomial time bound can be reduced to SAT [Coo71]. Idea:
  - TM configuration $\sim$ a valuation of propositional variables
  - sequence of configurations $\sim$ sequence of valuations
  - relations between consecutive configurations $\sim$ propositional formula
  - initial and accepting configurations $\sim$ propositional formula
  - accepting computation $\sim$ valuation that makes the formula true
- The proof is similar to the reduction from AI planning to SAT! We will discuss the topic in detail later.

Significance of NP-completeness

- No NP-complete problem is known to have a polynomial time algorithm.
- Best algorithms have a worst-case exponential runtime.

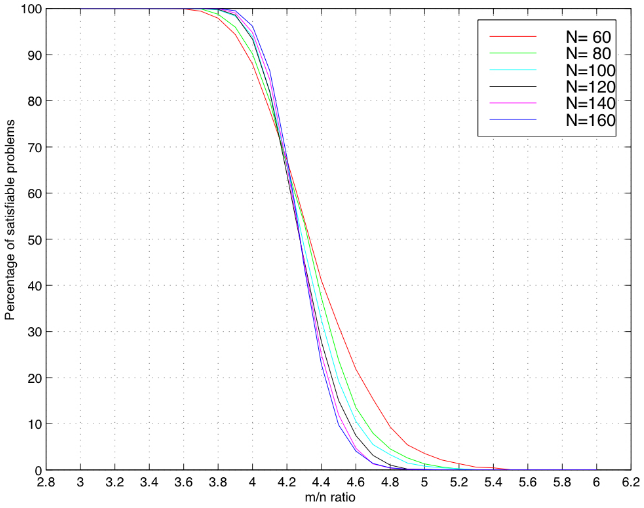
  $2^{0.30897m}$, $2^{0.10299L}$ [Hir00]

  $(2 - \frac{2}{k+1})^n$ [DGH+02]

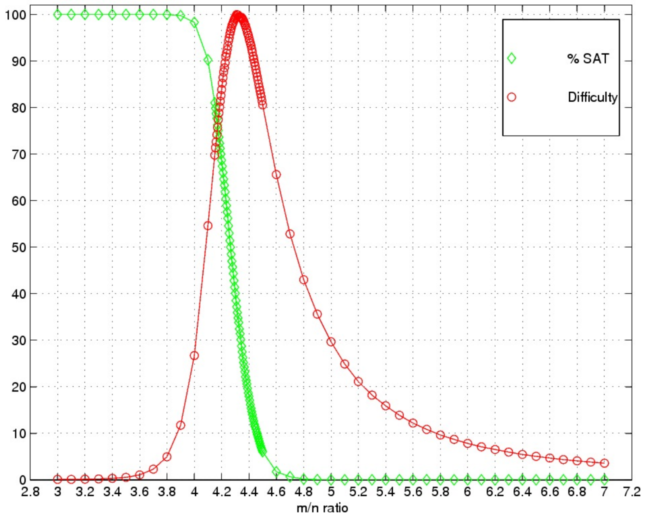
  $2^{n(1-\frac{1}{\ln \frac{m}{n} + O(\ln \ln m)})}$ [DHW05]

  (m clauses of length $\leq k$, n variables, size L).
- However, worst-case doesn't always show up!
- Current SAT algorithms can solve problem instances with millions of clauses and hundreds of thousands of variables in seconds.

Phase transitions
phase transition from SAT to UNSAT in 3-SAT

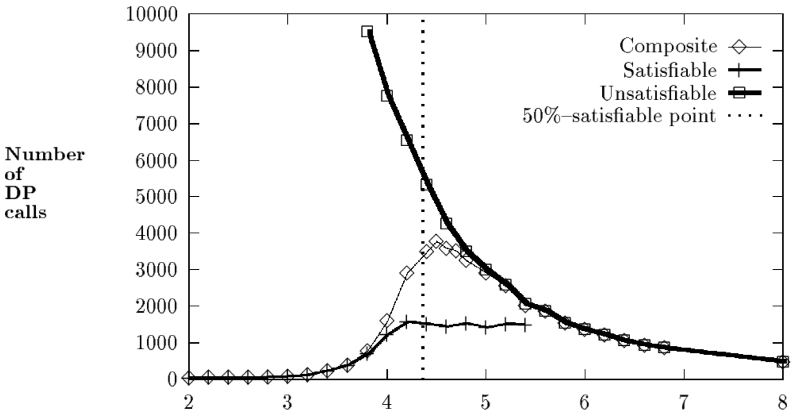


Phase transitions
Problem difficulty in the phase transition area



Phase transitions

Problem difficulty separately for SAT and UNSAT



Meaning of phase transitions

Even though all known complete algorithms have an exponential runtime in the worst case, their scalability on under-constrained and over-constrained problem instances is often much much better.

Other hard problems have similar phase transitions: keep problem size constant, and vary one of the parameters.

- scheduling: few..many tasks, a lot of..little time
- diagnosis: few..many observations
- planning, model-checking: many..few transitions

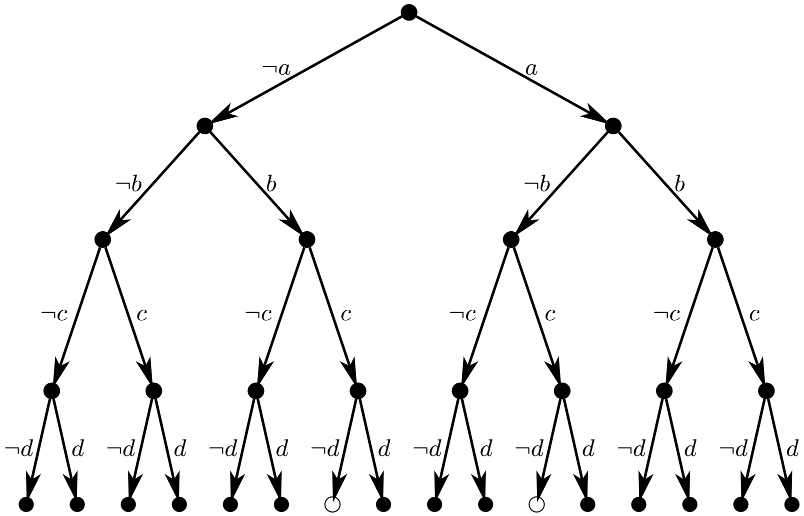
Truth-tables

Truth table for
$\phi = (a \leftrightarrow b) \vee (c \rightarrow d)$:

v				$v(\phi)$
a	b	c	d	
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Truth-tables vs binary search trees

Binary search tree for $\phi = (a \leftrightarrow b) \vee (c \rightarrow d)$:



The Resolution Rule

Resolution

$$\frac{l \vee \phi \qquad \bar{l} \vee \phi'}{\phi \vee \phi'}$$

One of l and $\bar{l}$ is false.
Hence at least one of ϕ and ϕ' is true.

Unit Resolution

- Unrestricted application of the resolution rule is too expensive.
- Unit resolution restricts one of the clauses to be a unit clause consisting of only one literal.
- Performing all possible unit resolution steps on a clause set can be done in linear time [DG84], and there are very efficient implementations [MMZ^{+}01].

Unit Propagation

Unit Resolution

$$\frac{l \qquad \bar{l} \vee \phi}{\phi}$$

Unit Propagation algorithm UNIT($\mathcal{F}$) for clause sets $\mathcal{F}$

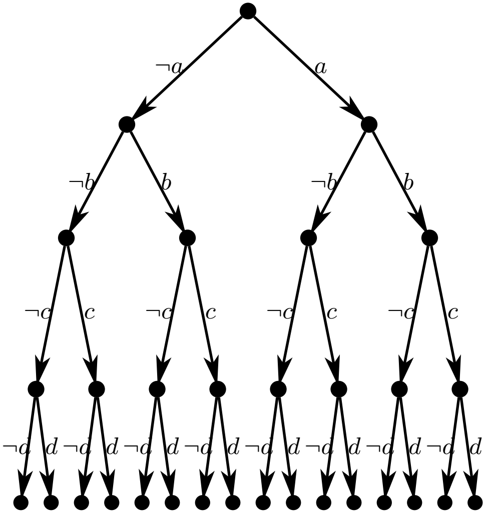
1. If there is a unit clause $l \in \mathcal{F}$, then replace every $\bar{l} \vee \phi \in \mathcal{F}$ by ϕ and remove all clauses containing l from $\mathcal{F}$.
   As a special case the empty clause $\perp$ may be obtained.
2. If $\mathcal{F}$ still contains a unit clause, repeat step 1.
3. Return $\mathcal{F}$.

We sometimes write $\mathcal{F} \vdash_{UP} l$ if $l \in UP(\mathcal{F})$.

Binary search with unit resolution

The Davis-Putnam-Logemann-Loveland procedure DPLL [DLL62]

$a \vee d$
$b \vee c$



Binary search with unit resolution

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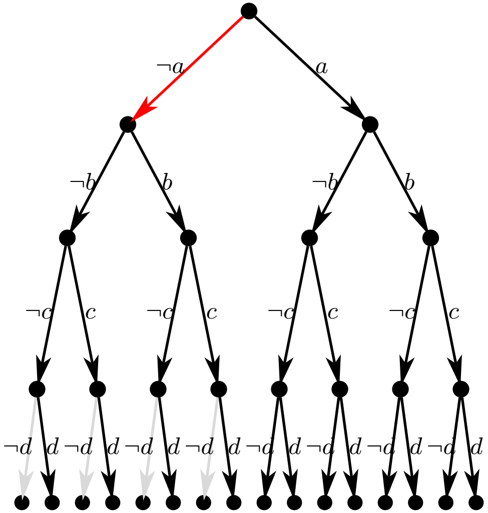

Binary search with unit resolution

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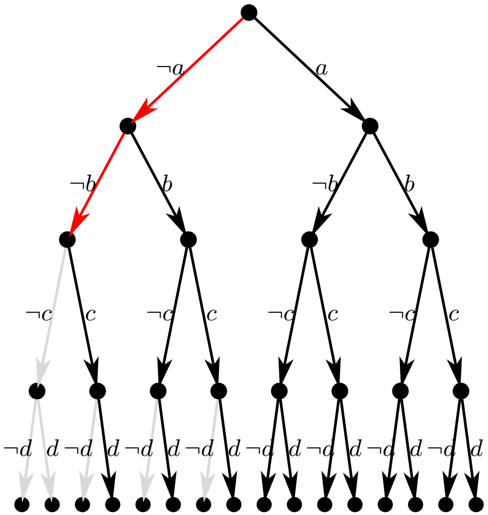

$\neg a$
a
$\neg b$
b
$\neg b$
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$\neg c$
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Davis-Putnam-Logemann-Loveland procedure
[DLL62]

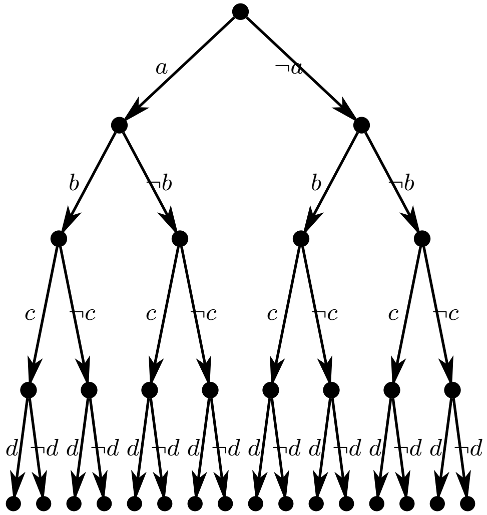
```
1: PROCEDURE DPLL($C$)
2: $C$ := UNIT$(C)$;
3: IF $\{x, \neg x\} \subseteq C$ for some $x \in X$ THEN RETURN false;
4: $x$ := any variable such that $\{x, \neg x\} \cap C = \emptyset$;
5: IF no such variable exists THEN RETURN true;
6: IF DPLL($C \cup \{x\}$) = true THEN RETURN true;
7: RETURN DPLL($C \cup \{\neg x\}$);
```

DPLL with backjumping

- The DPLL backtracking procedure often discovers the same conflicts repeatedly.
- In a branch $l_1, l_2, \dots, l_{n-1}, l_n$, after l_n and $\overline{l_n}$ have led to conflicts (derivation of $\bot$), $\overline{l_{n-1}}$ is always tried next, even when it is irrelevant to the conflicts with l_n and $\overline{l_n}$.
- Backjumping [Gas77] can be adapted to DPLL to backtrack from l_n to l_i when $l_{i+1}, \dots, l_{n-1}$ are all irrelevant.

DPLL with backjumping

$\neg a \vee b$
$\neg b \vee \neg d \vee e$
$\neg d \vee \neg e$
$\neg b \vee d \vee e$
$d \vee \neg e$
$c \vee f$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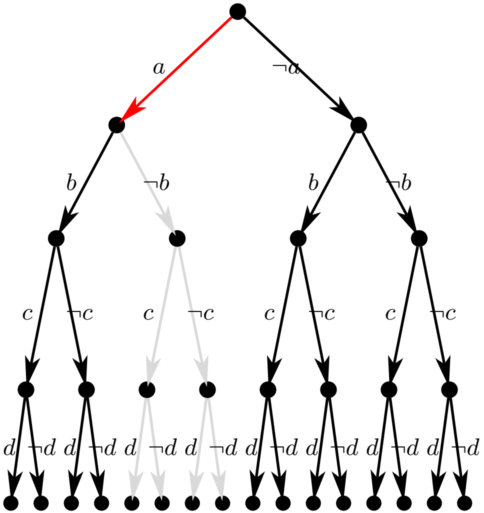
DPLL with backjumping

$\neg a \vee b$
$\neg b \vee \neg d \vee e$
$\neg d \vee \neg e$
$\neg b \vee d \vee e$
$d \vee \neg e$
$c \vee f$



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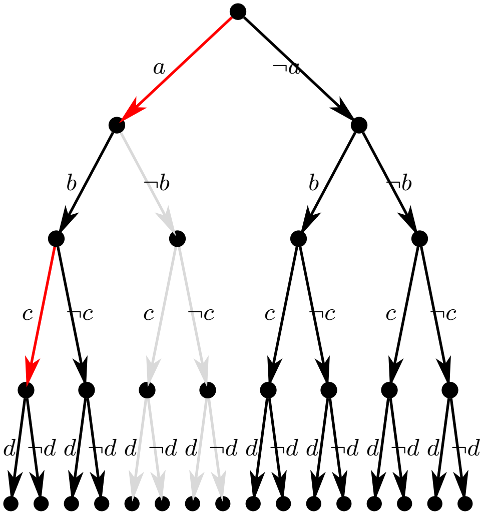
DPLL with backjumping

$\neg a \lor b$
$\neg b \lor \neg d \lor e$
$\neg d \lor \neg e$
$\neg b \lor d \lor e$
$d \lor \neg e$
$c \lor f$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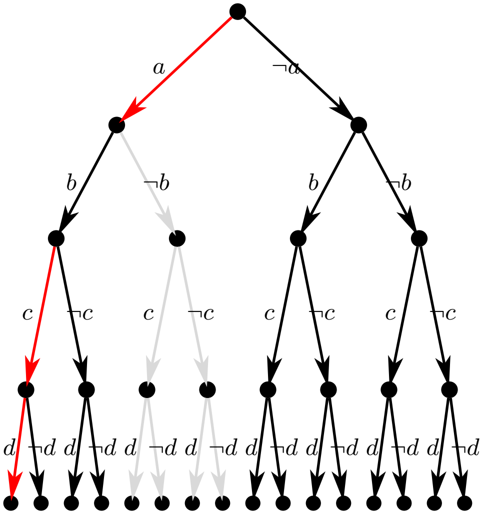
DPLL with backjumping

$\neg a \vee b$
$\neg b \vee \neg d \vee e$
$\neg d \vee \neg e$
$\neg b \vee d \vee e$
$d \vee \neg e$
$c \vee f$

Conflict set with d: $\{a, d\}$



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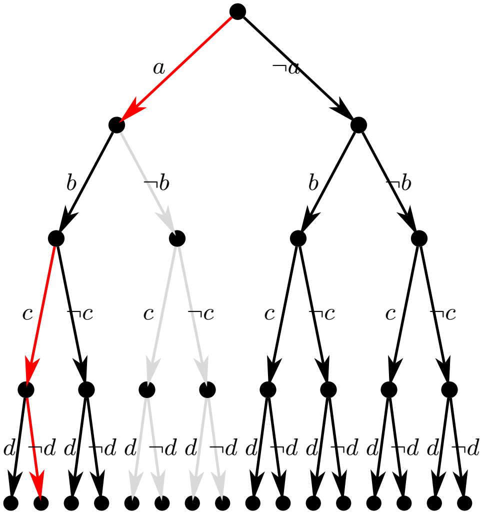
DPLL with backjumping

$\neg a \vee b$
$\neg b \vee \neg d \vee e$
$\neg d \vee \neg e$
$\neg b \vee d \vee e$
$d \vee \neg e$
$c \vee f$

Conflict set with d: $\{a, d\}$
Conflict set with $\neg d$: $\{a, \neg d\}$

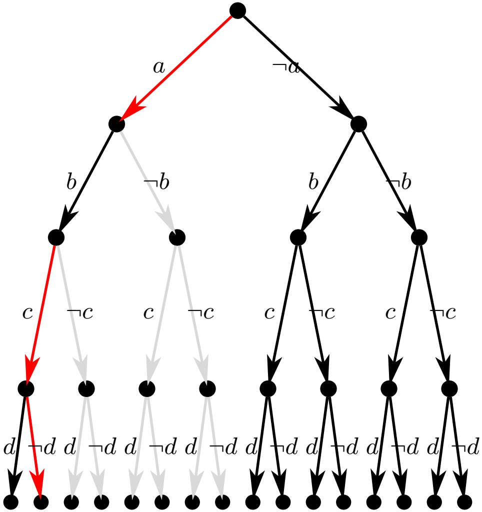


DPLL with backjumping

$$\neg a \vee b$$
$$\neg b \vee \neg d \vee e$$
$$\neg d \vee \neg e$$
$$\neg b \vee d \vee e$$
$$d \vee \neg e$$
$$c \vee f$$

Conflict set with d: $\{a, d\}$
Conflict set with $\neg d$: $\{a, \neg d\}$

No use trying $\neg c$.
Directly go to $\neg a$.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Conflict-Driven Clause Learning (CDCL)
[MSS96]

- The Resolution rule is more powerful than DPLL: UNSAT proofs by DPLL may be exponentially bigger than the smallest resolution proofs.
- An extension to DPLL, based on learned clauses, is similarly exponentially more powerful than DPLL [BKS04].
- It has been shown that CDCL with restarts is equally powerful to resolution [PD09a].
- In many applications, SAT solvers with CDCL are the best.

Conflict-Driven Clause Learning (CDCL)

- Assume a partial valuation (a path in the DPLL search tree from the root to a leaf node) corresponding to literals $l_1, \dots, l_n$ leads to a contradiction (with unit resolution)

  $$\mathcal{F} \cup \{l_1, \dots, l_n\} \vdash_{UP} \bot$$

  From this follows

  $$\mathcal{F} \models \overline{l_1} \vee \cdots \vee \overline{l_n}.$$

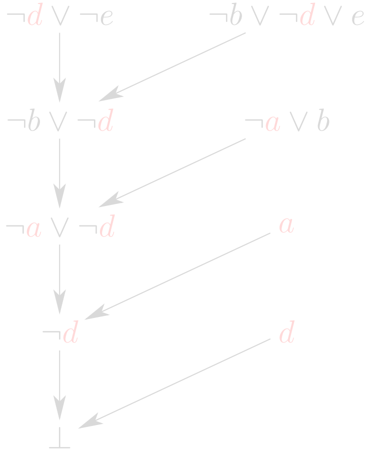
- Often not all of the literals $l_1, \dots, l_n$ are needed for deriving the empty clause $\bot$, and a shorter clause can be derived.

Conflict-Driven Clause Learning (CDCL)
Example

$\neg a \vee b$

$\neg b \vee \neg d \vee e$

$\neg d \vee \neg e$ falsified



$$\begin{tikzcd}
\neg d \vee \neg e \arrow[d] & \neg b \vee \neg d \vee e \arrow[dl] \\
\neg b \vee \neg d \arrow[d] & \neg a \vee b \arrow[dl] \\
\neg a \vee \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\perp &
\end{tikzcd}$$

Conflict-Driven Clause Learning (CDCL)

Example

$\neg a \vee b$

$\neg b \vee \neg d \vee e$

$\neg d \vee \neg e$ falsified

$$\begin{tikzcd}
\neg d \vee \neg e \arrow[d] & \neg b \vee \neg d \vee e \arrow[dl] \\
\neg b \vee \neg d \arrow[d] & \neg a \vee b \arrow[dl] \\
\neg a \vee \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\perp &
\end{tikzcd}$$

a, b

Conflict-Driven Clause Learning (CDCL)

Example

$$\neg a \vee b$$

$$\neg b \vee \neg d \vee e$$

$$\neg d \vee \neg e$$ falsified

$$a, b, c$$

$$\begin{tikzcd}
\neg d \vee \neg e \arrow[d] & \neg b \vee \neg d \vee e \arrow[dl] \\
\neg b \vee \neg d \arrow[d] & \neg a \vee b \arrow[dl] \\
\neg a \vee \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\perp &
\end{tikzcd}$$

Conflict-Driven Clause Learning (CDCL)

Example

$\neg a \vee b$

$\neg b \vee \neg d \vee e$

$\neg d \vee \neg e$ falsified

a, b, c, d, e

$$\begin{tikzcd}
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\neg a \vee \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\bot &
\end{tikzcd}$$

Conflict-Driven Clause Learning (CDCL)

Example

$\neg a \vee b$

$\neg b \vee \neg d \vee e$

$\neg d \vee \neg e$ falsified

a, b, c, d, e

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\neg a \vee \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\bot &
\end{tikzcd}$$

Conflict-Driven Clause Learning (CDCL)

Example

$$\neg a \lor b$$

$$\neg b \lor \neg d \lor e$$

$$\neg d \lor \neg e$$ falsified

$$a, b, c, d, e$$

$$\begin{tikzcd}
\neg d \lor \neg e \arrow[d] & \neg b \lor \neg d \lor e \arrow[dl] \\
\neg b \lor \neg d \arrow[d] & \neg a \lor b \arrow[dl] \\
\neg a \lor \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\bot &
\end{tikzcd}$$

Conflict-Driven Clause Learning (CDCL)
Example

$\neg a \lor b$

$\neg b \lor \neg d \lor e$

$\neg d \lor \neg e$ falsified

a, b, c, d, e

$$\begin{tikzcd}
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\neg a \lor \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
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Conflict-Driven Clause Learning (CDCL)

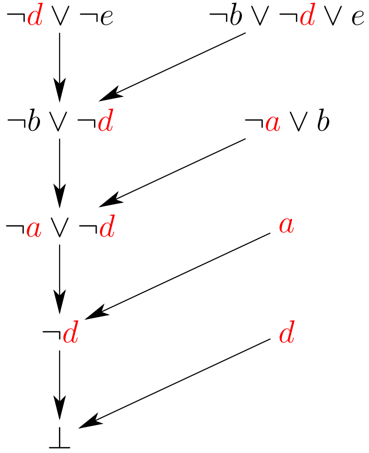
Example

$\neg a \vee b$

$\neg b \vee \neg d \vee e$

$\neg d \vee \neg e$ falsified

a, b, c, d, e



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\neg b \vee \neg d \arrow[d] & \neg a \vee b \arrow[dl] \\
\neg a \vee \neg d \arrow[d] & a \arrow[dl] \\
\neg d \arrow[d] & d \arrow[dl] \\
\bot &
\end{tikzcd}$$

Conflict-Driven Clause Learning (CDCL)
Procedure

The Reason of a Literal

For each non-decision literal l a reason is recorded: it is the clause $l \vee \phi$ from which it was derived with $\neg\phi$.

A Basic Clause Learning Procedure

- Start with the clause $C = l_1 \vee \cdots \vee l_n$ that was falsified.
- Resolve it with the reasons $\bar{l} \vee \phi$ of non-decision literals $\bar{l}$ until only decision variables are left.

Conflict-Driven Clause Learning (CDCL)
Different variants of the procedure

decision scheme	Stop when only decision variables left.
First UIP	(Unique Implication Point) Stop when only one literal of current decision level left.
Last UIP	Stop when at the current decision level only the decision literal is left.

First UIP is usually considered to be the most useful.
Some solvers learn multiple clauses.

Conflict-Driven Clause Learning (CDCL)

Forgetting/deleting clauses

- In contrast to DPLL, a main problem with CDCL is the high number of learned clauses.
- To avoid memory filling up, large numbers of learned clauses are deleted at regular intervals, typically based on clause length, last use, and other criteria.
- One interesting strategy is to rank the clauses according to the number of decision levels appearing in the clause [AS09].

Heuristics for CDCL: VSIDS
Variable State Independent Decaying Sum [MMZ+01]

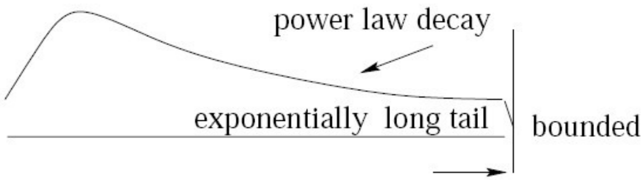
- Initially the score $s(l)$ of literal l is its number of occurrences in $\mathcal{F}$.
- When clause with l is learned, increase $r(l)$.
- Periodically decay the scores:

$$s(l) := r(l) + 0.5s(l); \qquad r(l) := 0;$$

- Always choose unassigned literal l with maximum $s(l)$.

Variations and extensions of VSIDS most popular in current solvers.

Heavy-tailed runtime distributions



(Diagram from [CGS01])

On many NP-complete problems, heavy-tailed distributions characterize

- runtimes of a randomized algorithm on a single instance and
- runtimes of a deterministic algorithm on a class of instances.

Heavy-tailed runtime distributions
Estimating the mean is problematic

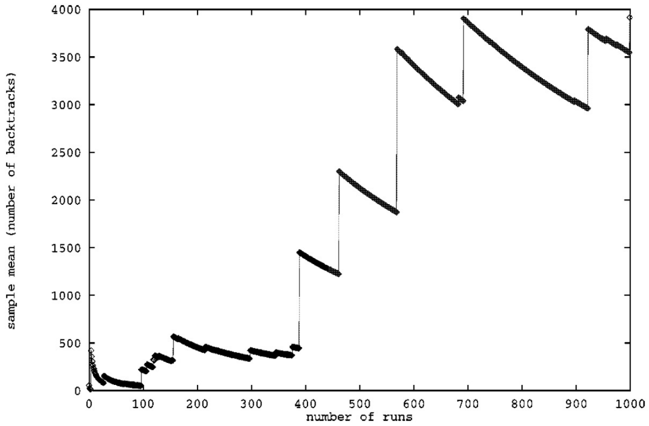


Diagram from [GSCK00]

Heavy-tailed runtime distributions
Cause

- A small number of wrong decisions lead to a part of the search tree not containing any solutions.
- Backtrack-style search needs a long time to traverse the search tree.
    - Many short paths from the root node to a success leaf node.
    - High probability of reaching a huge subtree with no solutions.

These properties mean that

- average runtime is high,
- restarting the procedure after t seconds reduces the mean substantially, if t is close to the mean of the original distribution.

Restarts in SAT algorithms
Answer to heavy-tailedness

Restarts had been used in stochastic local search algorithms:

- Necessary for escaping local minima!

Gomes et al. demonstrated the utility of restarts for systematic SAT solvers:

- Small amount of randomness in branching variable selection.
- Restart the algorithm after a given number of seconds.

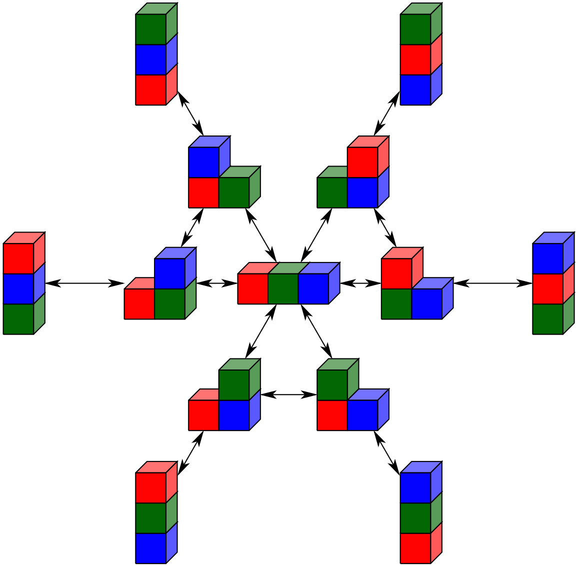
Restarts with CDCL

- Learned clauses are retained when doing the restart.
- Problem: Optimal restart policy depends on the runtime distribution, which is generally not known.
- Problem: Deletion of learned clauses and too early restarts may lead to non-termination for unsatisfiable formulas. This is avoided by gradually increasing restart interval.
- One effective restart strategy is based on the Luby series $n = 1, 1, 2, 1, 1, 2, 4, 1, 1, 2, 1, 1, 2, 4, 8, ...$, learning e.g. $30n$ clauses between consecutive restarts [Hua07].

Application: Reachability

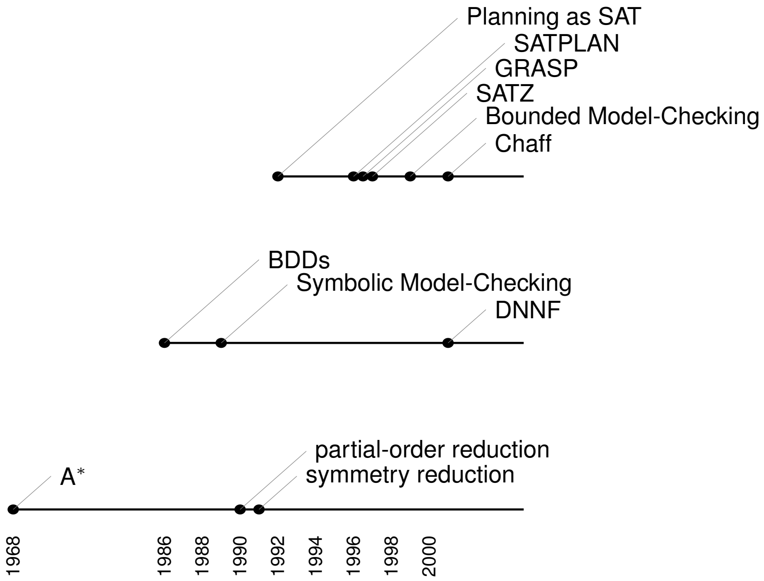
- finding a path from a state from in I to a state in set G in a succinctly/compactly represented graph
- PSPACE-complete [GW83, Loz88, LB90, Byl94]
- in NP when restricted to paths of polynomial length
- Basis of efficient solutions to
  - planning problem in AI [KS92, KS96]
  - LTL model-checking problem [BCCZ99]
  - DES diagnosis problem [GARK07]
- Often replacing traditional state-space search methods
- One of the first and most prominent applications of SAT
- Extensions to timed systems with SAT modulo Theories (SMT)

State-space transition graphs
Blocks world with three blocks



State-space search and satisfiability

Explicit state-space search; symbolic search with BDDs, SAT



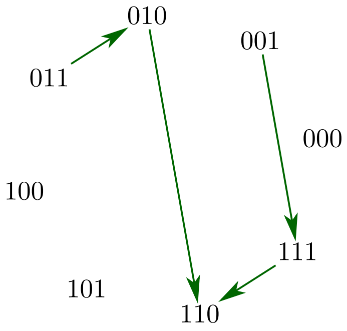
Transition relations in propositional logic

State variables are
$X = \{a, b, c\}$.

$$(\neg a \wedge b \wedge c \wedge \neg a' \wedge b' \wedge \neg c') \vee$$
$$(\neg a \wedge b \wedge \neg c \wedge a' \wedge b' \wedge \neg c') \vee$$
$$(\neg a \wedge \neg b \wedge c \wedge a' \wedge b' \wedge c') \vee$$
$$(a \wedge b \wedge c \wedge a' \wedge b' \wedge \neg c')$$

The corresponding matrix is

	000	001	010	011	100	101	110	111
000	0	0	0	0	0	0	0	0
001	0	0	0	0	0	0	0	1
010	0	0	0	0	0	0	1	0
011	0	0	1	0	0	0	0	0
100	0	0	0	0	0	0	0	0
101	0	0	0	0	0	0	0	0
110	0	0	0	0	0	0	0	0
111	0	0	0	0	0	0	1	0



Transition relations in propositional logic

Let $X = \{x_1, \dots, x_n\}$ be the state variables.

- Any deterministic action/event corresponds to a partial function.
  Partial functions correspond to conjunctions of a precondition formula $\Pi(x_1, \dots, x_n)$ and equivalences

  $$x'_i \leftrightarrow F(x_1, \dots, x_n)$$

  for every $x_i \in X$.
- Choice between actions/events $\alpha_1, \dots, \alpha_k$ corresponds to

  $$\Phi = \alpha_1 \vee \dots \vee \alpha_k.$$

Reachability as SAT

Let $\Phi(n, m)$ denote the formula obtained from Φ by replacing each $x \in X$ by $x@n$ and each x' by $x@m$.
Then satisfying valuations of

$$\Phi(0,1) \wedge \Phi(1,2) \wedge \cdots \Phi(n-1,n)$$

are in 1-to-1 correspondence to paths of length n in the transition graph.

Testing whether a state satisfying G can be reached from a state satisfying I in n steps reduces to testing the satisfiability of

$$I(0) \wedge \Phi(0,1) \wedge \Phi(1,2) \wedge \cdots \Phi(n-1,n) \wedge G(n).$$

Applications

Interpretations of SAT tests

$$I(0) \wedge \Phi(0,1) \wedge \Phi(1,2) \wedge \cdots \Phi(n-1,n) \wedge G(n).$$

Planning Can goals G be reached from the initial state I [KS96]?

Model-checking Can the safety property $\neg G$ be violated on executions that start from I? (Extensions for LTL model-checking in [BCCZ99].)

DES Diagnosis Consider

$$\Phi(0,1) \wedge \Phi(1,2) \wedge \cdots \Phi(n-1,n) \wedge (o_1@t_1 \wedge \cdots \wedge o_m@t_m) \wedge F.$$

Are observations $o_1, \ldots, o_m$ respectively at $t_1, \ldots, t_m$ compatible with fault assumptions F [GARK07]? F encodes e.g. “there are n faults between time points 0 and n.

Applications

The most basic encodings given above can often be improved.

- optimal (linear-size) encodings [LBHJ04, RHN06]
- multiple actions in parallel [RHN06]
- scheduling the SAT tests for different path lengths [Rin04, Zar04] in parallel
- search heuristics replacing VSIDS [Gan11, Rin10, Rin12b]
- reachability-specific implementation technology [Rin12a]

MAXSAT
Motivation

- Many AI problems involve optimization:
  - Learn an explanation with the best match to data [Cus08].
  - Find a least-cost plan [RGPS10].
  - Select best drugs for cancer therapy [LK12].
- SAT insufficient: answers a binary yes–no question
- MAXSAT extends SAT with a basic form of optimization.
- Other frameworks: Mixed Integer-Linear Programming (MILP/ILP/MIP), constraint programming and optimization [DRGN10], SMT + optimization [ST12]
- advantage over MILP: efficient Boolean reasoning

Introduction: (Weighted) (Partial) MAXSAT

plain MAXSAT	Maximize the number of satisfied clauses
partial MAXSAT	Maximize the number of satisfied soft clauses Hard clauses must be satisfied
weighted MAXSAT	Maximize the sum of weights of satisfied clauses

Decision problem “is there an valuation with weight $\geq n$” NP-complete.

The FP^{NP} optimization problem solvable by a polynomial number of SAT calls.

Algorithms for MAXSAT

- reduction to a sequence of SAT problems [FM06, ABL13, DB11]
- branch and bound [HLO08, LMMP10]
- Mixed Integer Linear Programming [DB13] (CPLEX)

Some MAXSAT solvers

dfs + bounding	MaxSatz, MiniMaxSat
SAT	sat4j, wbo, wpm, pwbo, maxhs

MAXSAT by a sequence of SAT queries

1. From a weighted partial MAXSAT instance, construct a SAT instance [FM06, ABL13]:
   - Hard clauses are taken as is.
   - For each soft clause $l_1 \vee \cdots \vee l_n$, have $b \vee l_1 \vee \cdots \vee l_n$, where b is a new auxiliary variable.
2. If the SAT instance is unsatisfiable, the best valuation so far is the globally best (And if this was the first time here, the hard clauses are unsatisfiable.)
3. Otherwise, each true b variable corresponds to a (possibly) false soft clause.
4. Calculate the sum F of the weights of true soft clauses.
5. Construct a new SAT instance, with cardinality constraints [BB03, Sin05] requiring that weights of true soft clauses $> F$.
6. One can also add a clause requiring at least one previously false soft clause to be true. (unsatisfiable cores [ABL13])
7. Continue from step 2.

Other query strategies

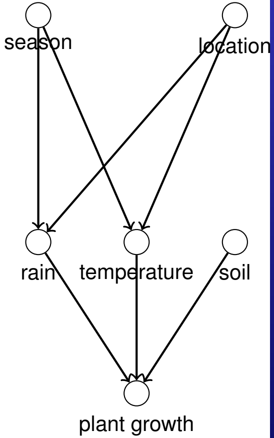
Given SAT instances saying “at most k soft clauses are false”, alternative query strategies are possible.

- unsatisfiability based: try $k = 0$, then $k = 1$, and so on.
- satisfiability based: try $k = k_{max} - 1$, then $k = k_{max} - 2$, and so on.
- binary search: try half-way between 0 and k_{max}, and after tightening either lower or upper bound, then again half-way.

Same question of SAT queries with different parameter values k arises also in other SAT and constraints applications, including planning and scheduling, with other algorithms proposed [Rin04, SS07]. (Usefulness of these algorithms to MAXSAT is not clear.)

Bayesian networks

- Compact representation of probability distributions [Pea89]
- Makes probabilistic dependence and independence explicit.
- lots of applications e.g. in intelligent robotics, especially for dynamic Bayesian networks
- Other graphical models: Markov networks [Pea89]



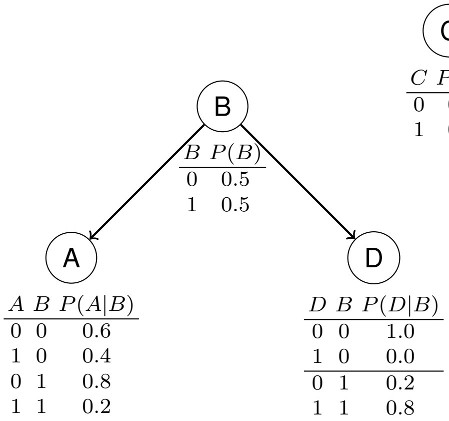
Bayesian networks

- Probabilistic Inference (PI): calculate marginal probability of a variable given evidence
- Most Probable Explanation (MPE): find a valuation for the variables with the highest probability
- Maximum A Posteriori hypothesis (MAP) [PD04]: find hypotheses that explain the observations best
- Structure Learning (SL): find Bayesian network that best matches given data

problem	complexity	SAT variant
PI	#P	#SAT
MPE	FP^{NP}	MAXSAT
MAP	NP^{PP}	E-MAJSAT (SSAT)
SL	FP^{NP}	MAXSAT

MPE: Most Probable Explanation

- Of all valuations of the variables, find one with the highest probability.
- Has the flavor of diagnosis problems (but see the MAP problem later!)
- Solution e.g. by reduction to MAXSAT [KD99, Par02]



C	$P(C)$
0	0.9
1	0.1

B	$P(B)$
0	0.5
1	0.5

A	B	$P(A\|B)$
0	0	0.6
1	0	0.4
0	1	0.8
1	1	0.2

D	B	$P(D\|B)$
0	0	1.0
1	0	0.0
0	1	0.2
1	1	0.8

Reduction of MPE to MAXSAT

A	B	$P(A\|B)$
0	0	0.6
1	0	0.4
0	1	0.8
1	1	0.2

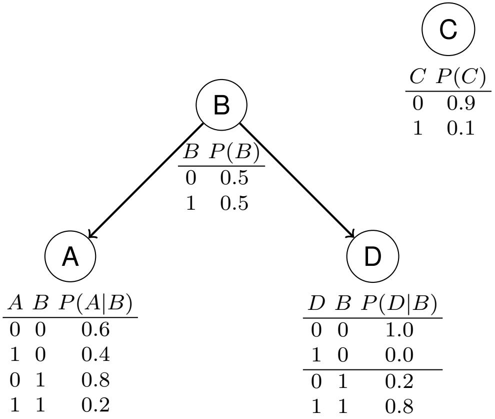
translates to

$\neg A \wedge \neg B$	probability 0.6
$\neg A \wedge B$	probability 0.4
$A \wedge \neg B$	probability 0.8
$A \wedge B$	probability 0.2

- Problem 1: Probabilities must be multiplied to get the overall probability.
- Solution: Sum the logarithms of the probabilities.
- Problem 2: Probabilities 0 correspond to $\log 0 = \infty$.
- Solution: Use hard clauses.
- Negate the conjunctions to get clauses.
  Negate $\log p$ (with $p \leq 1$) to get positive weights.

Structure Learning for Bayesian network

ABCD
0000
0101
1000
0101
0000
1101
0010
1000
0101
0100



B	$P(B)$
0	0.5
1	0.5

C	$P(C)$
0	0.9
1	0.1

A	B	$P(A\|B)$
0	0	0.6
1	0	0.4
0	1	0.8
1	1	0.2

D	B	$P(D\|B)$
0	0	1.0
1	0	0.0
0	1	0.2
1	1	0.8

Structure Learning for Bayesian networks
Mapping to Constraint Satisfaction, including MAXSAT

- The score of a network is the sum of all per-node scores.
- The score of each node is determined by its parents: each alternative parent set has a score.
- Constraint satisfaction formulation:
    - Choose a parent set for each node. (E.g. max. 3 parents)
    - The resulting graph must be acyclic.
    - Objective: maximize the sum of the parent set scores.
- main challenge in encoding: acyclicity constraint
    - transitive ancestor relation [Cus08]
    - total ordering of nodes [Cus08]
    - recursively define distance from leaf 0, 1, 2, ...

Structure Learning for Bayesian networks

- Finding optimal nets translatable into MAXSAT, MILP etc.
- Optimal solutions found for nets of up to some dozens of nodes.
- On many standard benchmarks, MAXSAT and MILP solvers comparable.
- Best methods enhance MILP with specialized heuristics [Cus11].

Methods used for approximate solutions are different!

Model-Counting (#SAT)

- How many satisfying valuations does a propositional formula have?
- The problem is #P-complete [Val79].
- Interestingly, model-counting is #P-complete also when SAT is easy (in P): DNF-SAT, 2-SAT, Horn-SAT, ... [Val79].
- #P harder than NP: $\phi \in$SAT if and only if model-count ≥ 1

Weighted Model-Counting

- Weighted Model-Counting assigns a weight to each literal.
- Compute the sum of the weights of satisfying valuations.
- Weight of a valuation is the product of weights of true literals.
- This generalization is useful e.g. for probabilistic reasoning.
- Coincides with unweighted MC when all weights are 1.

Algorithms for Model Counting

- exact algorithms: extensions of DPLL and CDCL [BDP03, BDP09, SBB$^+$04, SBK05a, GSS09]
- approximate counting (upper bound)
- approximate counting (no guaranteed lower or upper bound) [KSS11]

Algorithms
extensions of DPLL and CDCL

- basic algorithm: DPLL-style tree search
- connected components [BP00]
- component caching [BDP03]
- combining clause-learning with component caching [SBB$^+$04]
- heuristics [SBK05a]

Basic model-counting DPLL algorithm

Consider a model-counting run of DPLL for a formula with propositional variables X.

- Two branches $\{x\} \cup C$ and $\{\neg x\} \cup C$ disjoint $\Longrightarrow$ take the sum the respective model counts.
- When DPLL detects that all clauses are satisfied with n variables assigned, the count for the branch is

$$2^{|X|-n}$$

Component analysis and component caching

Enhancements to the basic model-counting DPLL (e.g. in Cachet [SBB$^+$04]):

- Component analysis: if C can be partitioned to $(C_1, \ldots, C_n)$ so that partitions don't share variables, then count each C_i separately and take the product of the counts [BP00]
- Component caching [BDP03]: record model-counts and recall them when encountering a clause set again.

Efficient model-counts for normal forms

- Model-counting for CNF (#SAT) is #P-complete [Val79].
- Some normal forms have polynomial time model-counting.
  - Binary Decision Diagrams (BDD) [Bry92]
  - deterministic Decomposable Negation Normal Form (d-DNNF) [Dar02]
- Reaching these normal forms can take exponential time, space.
- Some of the best translators for these normal forms [HD07] are similar to the model-counting variants of the Davis-Putnam procedure, for example in utilizing component analysis.

MC Applications: Bayesian inference

- optimal distinguishing tests [HS09]
- Bayesian inference [BDP09, SBK05b, CD08], calculating marginal probabilities of some variables given values of other variables of a Bayesian network.
  (There are interesting connections between specialized Bayesian inference algorithms and model-counting algorithms. E.g., many can be viewed as instances of algorithms for the SumProd problem [BDP09].)

Probabilistic Inference by Model-Counting

Marginal probability of given evidence

$$\begin{tikzcd}
C & & B \arrow[dl] \arrow[dr] & \\
& A & & D
\end{tikzcd}$$

C	$P(C)$
0	0.9
1	0.1

B	$P(B)$
0	0.5
1	0.5

A	B	$P(A\|B)$
0	0	0.6
1	0	0.4
0	1	0.8
1	1	0.2

D	B	$P(D\|B)$
0	0	1.0
1	0	0.0
0	1	0.2
1	1	0.8

- Variable for each node A, B, C, D.
- Parentless nodes have the obvious weights $w(B) = w(\neg B) = 0.5$, $w(C) = 0.1, w(\neg C) = 0.9$.
- Chance variables $c_{A|B}$ and $c_{A|\neg B}$ for nodes with parents.

$$\begin{array}{cc}
w(c_{A|B}) = 0.2 & w(\neg c_{A|B}) = 0.8 \\
w(c_{A|\neg B}) = 0.4 & w(\neg c_{A|\neg B}) = 0.6 \\
w(A) = 1 & w(A) = 1
\end{array}$$

- $B \wedge c_{A|B} \rightarrow A$
  $B \wedge \neg c_{A|B} \rightarrow \neg A$
  $\neg B \wedge c_{A|\neg B} \rightarrow A$
  $\neg B \wedge \neg c_{A|\neg B} \rightarrow \neg A$
- Conditioning with evidence $B, \neg C$ by adding in the clause set.

Stochastic Satisfiability SSAT

- Stochastic satisfiability [Pap85] extends propositional logic with stochastic AND-OR quantification. (An extension of Quantified Boolean formulas (QBF) [Sto76]).
- Prefix consisting of variables quantified by $\exists$, $\forall$ and $\reflectbox{R}^r$, followed by a propositional formula.

In SSAT, the probability $P(\phi)$ associated with a formula ϕ is defined recursively as follows.

- Base case: variable free (quantifier free) formulas containing only atomic formulas $\perp$ and $\top$ and Boolean connectives.
  $P(\top) = 1.0$
  $P(\perp) = 0.0$
- $P(\exists x \phi) = \max(P(\phi[\top/x]), P(\phi[\perp/x]))$
- $P(\reflectbox{R}^r x \phi) = r \times P(\phi[\top/x]) + (1 - r) \times P(\phi[\perp/x])$
- $P(\forall x \phi) = \min(P(\phi[\top/x]), P(\phi[\perp/x]))$

Question: Is $P(\phi) \geq R$ for some $R \in [0, 1[$?

Stochastic Satisfiability SSAT
Special cases

SSAT can be viewed as a generalization of

- SAT: quanfiers $\exists$ only
- TAUT: quanfiers $\forall$ only
- quantified Boolean formulas (QBF): quantifiers $\exists$, $\forall$ only [Sto76]
- E-MAJSAT: prefix $\exists\exists\cdots\exists\text{ꓤ}^{r_1}\text{ꓤ}^{r_2}\cdots\text{ꓤ}^{r_n}$ [PD09b]

Algorithms for E-MAJSAT and SSAT

- Basic approach [Lit99, LMP01]:
  - DPLL-style tree search
  - variables selected in quantification order
  - prune subtrees if irrelevant for establishing the lb R (thresholding [ML03])
  - component caching (as in model-counting #SAT)
- Implementations reported by Majercik, Littman, Boots [ML03, MB05].
- resolution rule [TF10] (following QBF resolution [KBKF95])
- SMT-style extension to cover the orthogonal problem of combining SAT with linear arithmetics (SSMT [TEF11])

Applications

- Maximum A Posteriori Hypothesis (MAP) is NP^{PP}-complete [PD04], corresponding to E-MAJSAT $(\exists \cdots \rotatebox[origin=c]{180}{R}^{r} \cdots)$
- MAP application: diagnosis
- Probabilistic verification of safety critical systems: what is the probability that event x will take place? [TF11]
- probabilistic planning [ML03]

MAP: Maximum A Posteriori Hypothesis

- MPE finds a single most probable valuation of variables.
- The probability of this valuation is typically low, and it is often not representative of the most likely fault e.g. in diagnosis.
- The Maximum A Posteriori Hypothesis (MAP) problem [PD04]:
  Find a valuation to a subset of hypothesis variables H that maximizes the probability of the given observations.
- Decision version of MAP is NP^{PP}-complete: guess a valuation of H; then verify that the probability of the observations is $\geq r$ for a given bound r.

MAP: Maximum A Posteriori Hypothesis
Encoding as E-MAJSAT

- How to choose hypotheses $h_1, \ldots, h_n$ to maximize the probability expressed by $\exists x_1 \exists x_2 \cdots \exists x_n \rotatebox[origin=c]{180}{R}^{0.5} y_1 \cdots \rotatebox[origin=c]{180}{R}^{0.5} y_m \phi$
- Encoding like Probabilistic Inference with Model-Checking. Difference is quantification:

$$\exists h_1 \exists h_2 \cdots \exists h_n \rotatebox[origin=c]{180}{R}^{w_1} x_1 \cdots \rotatebox[origin=c]{180}{R}^{w_m} x_m \Phi$$

where $x_1, ..., x_m$ are all the non-hypothesis variables with the same weights $w_1, \ldots, w_m$ as in the Probabilistic Inference problem.

Probabilistic planning by SSAT
[ML98]

$$\exists P \text{Я}^q C \exists E \left(I^0 \rightarrow \left(\bigwedge_{i=0}^{t-1} \mathcal{T}(i, i+1) \wedge G^t \right) \right) \qquad (1)$$

1. 1st block: $\exists$-quantification over all action sequences
2. 2nd block: Я-quantification over all contingencies
3. 3rd block: $\exists$-quantification over all executions of the plan

SMT: Satisfiability Modulo Theories

- numbers needed in representing
  - time
  - space (distance, size, ...)
  - resources (money, materials, ...)
- SAT has no numbers: reduction to SAT is feasible only for small integers
- SAT modulo Theories = SAT + specialized solvers for specific theories, such as
  - linear integer/rational/real arithmetic
  - bitvectors
  - graphs
- Similar to constraint programming frameworks.

Basic ideas of SMT

- Not everything is compactly expressible and efficiently solvable if only Boolean variables are used, for example real and rational arithmetics.
- SAT can be extended with non-Boolean theories. A clause has the form

$$l_1 \vee \cdots \vee l_n \vee E$$

  where E is a set of quantifier-free inequations over some set V of real/rational/other variables.
- The theories can be e.g.
  - linear inequalities,
  - mixed integer integer linear programs, or
  - something completely different.
- Compare: mixed integer linear programming MILP

SMT: Algorithms
Implementation

Extension of DPLL to theories

1. Run DPLL ignoring the inequations in the clauses.
2. After all Boolean variables have been set (at a leaf of the DPLL search tree), take the inequations $E_1, \ldots, E_m$ from all clauses that have no true literal.
3. Test with a specialized solver if $E_1 \cup \cdots \cup E_m$ is solvable. If it is, terminate.
4. Otherwise backtrack with the DPLL algorithm.

- The general idea is easy to implement for different theories, e.g. linear arithmetic.
- Early pruning of the DPLL search tree can be achieved by running the arithmetic solver before all Boolean variables are set.

SMT applications

- reachability with numeric state variables: planning with resources [WW99]
- reachability for timed transition systems: model-checking of timed systems [ACKS02], planning [SD05])

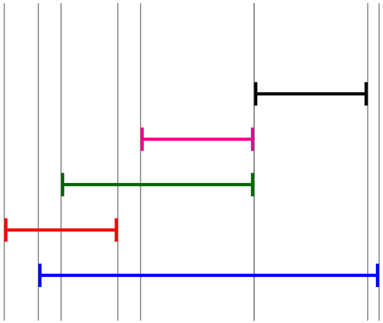
Timed systems reachability

- The most basic reachability problem (e.g. classical planning) is about instantaneous/asynchronous changes of (discrete) state variables.
- In timed systems, change may have a duration or a delay.
- Multiple simultaneous overlapping changes
- Change of continuous state variables may be continuous.
- Lots of applications: model-checking/verification of timed systems, temporal planning, temporal diagnosis, ...

SMT formalization of Timed Systems

Represent system state at time points where something non-continuous happens.

- Action is taken.
- Delayed effect of action takes place.
- A continuously changing variable reaches a critical value.



SMT formalization of Timed Systems

Actions and counters

Variable $\Delta@t$ indicates duration between time points $t-1$ and t.

Following is for actions a, state variables x, and counters C.

precondition of action	$a@t \rightarrow \phi@t$
counter initialization	$a@t \rightarrow (C@t = c)$
counter update	$\neg a@t \rightarrow (C@t = C@(t-1) - \Delta_t)$
discrete change	$(C@t = 0) \rightarrow x@t$
discrete change	$(C@t = 0) \rightarrow \neg x@t$
frame axiom	$(x@(t-1) \wedge \neg x@t) \rightarrow (C_1@t > 0 \vee \cdots$

Additionally, we need formulas to prevent overlap of actions using same resources.

SMT formalization of Timed Systems

Progress of time

Progress of time $\Delta@t$ between points $t-1$ and t.	
progress always positive	$\Delta@t > 0$
don't pass a scheduled change	$\Delta@t \leq C_k@(t-1)$

SMT applications

- Timed and hybrid systems analysis and verification [ABCS05]
- Planning in timed and hybrid systems [SD05]
- Timed and hybrid systems diagnosis:
  - Representation of observations: absolute time points
  - Representation of observations: temporal uncertainty

Other approaches to Timed Systems Reachability

- Explicit state-space search in the space of timed states (e.g. the UPPAAL model-checker [BLL$^+$96])
- Generate untimed transition sequences with SAT, then test whether possible to schedule [RGS13].
- Each method has strengths in different types of problems.

Conclusion
Algorithms

- NP-complete problems have become more solvable since mid-1990ies
- strength of algorithms such as CDCL over a wide range of SAT problems and applications
- convergence of search methods in different areas:
  - Probabilistic Inference for Bayesian networks vs. Model-Counting (#SAT)
  - reachability in AI planning and Computer Aided Verification
- increasing connections to combinatorial optimization methods, e.g. Mixed Integer Linear Programming

Conclusion
Problems

mappings complexity class - SAT variant - AI problem for reachability, planning, games:

NP	SAT	succinct reachability (poly-length paths)
NP	SMT	timed systems reachability (poly-length paths)
NP^{PP}	SSAT	succinct stochastic reachability (poly-length paths)
PSPACE	QBF	(succinct) 2-player games winning strategies
PSPACE	SSAT	stochastic 2-player games optimal strategies

probabilistic reasoning:

FP^{NP}	MAXSAT	Bayesian network MPE, SL
#P	#SAT	Bayesian network PI
NP^{PP}	E-MAJSAT	Bayesian network MAP

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